138. Proposed by HARRY S. VANDIVER, Bala, Pa.

Show that the number of solutions in positive integers for $x$, $y$, and $z$ of $x^3+2y^3+4z^3-6xyz=1$ is infinite.

Solution by the PROPOSER.

Suppose, for the present, that there is one solution of the proposed equation. Let $x=a$, $y=b$, $z=c$, satisfy it. Then

$$a^3+2b^3+4c^3-6abc=1. \quad \text{Whence } [a^3+2b^8+4c^3-6abc]^n=1.$$

Then whatever values of $x$, $y$, and $z$ we take consistent with $x^3+2y^3+4z^3-6xyz=1$, we will have

$$[a^3+2b^3+4c^3-6abc]^n=x^3+2y^3+4z^3-6xyz,$$

which is satisfied by the following assumptions:

$$\begin{aligned} &x+y\sqrt[3]{2}+z\sqrt[3]{4}=[a+b\sqrt[3]{2}+c\sqrt[3]{4}]^n \\ &x+\omega y\sqrt[3]{2}+\omega^2 z\sqrt[3]{4}=[a+\omega b\sqrt[3]{2}+\omega^2 c\sqrt[3]{4}]^n \quad (1) \\ &x+\omega^2 y\sqrt[3]{2}+\omega z\sqrt[3]{4}=[a+\omega^2 b\sqrt[3]{2}+\omega c\sqrt[3]{4}]^n \end{aligned}$$

(where $\omega^3=1$) as may be seen by multiplying together the equations, term for term. (1) gives on expanding

$$x+y\sqrt[3]{2}+z\sqrt[3]{4}=A+B\sqrt[3]{2}+C\sqrt[3]{4}$$

where $A$, $B$, and $C$ are positive integers.

Hence, we must have $x=A$, $y=B$, $z=C$.

(2) and (3) give likewise

$$\begin{aligned} &x+\omega y\sqrt[3]{2}+\omega^2 z\sqrt[3]{4}=A+\omega B\sqrt[3]{2}+\omega^2 C\sqrt[3]{4} \\ &x+\omega^2 y\sqrt[3]{2}+\omega z\sqrt[3]{4}=A+\omega^2 B\sqrt[3]{2}+\omega C\sqrt[3]{4} \end{aligned}$$

In each of these we will have as in (1) $x=A$, $y=B$, $z=C$.

Therefore, $x$, $y$, and $z$ have an infinite number of integral values depending on the value of $n$; provided, as we assumed at the start, that they have one set of values.

This one set of values is $x=1$, $y=1$, $z=1$; for $1^3+2\times3^3+4\times1^3-6=1$.

Let us find, for instance, the values of $x$, $y$, and $z$ for $n=2$, having given that $a=1$, $b=1$, $c=1$, we have

$$x+y\sqrt[3]{2}+z\sqrt[3]{4}=[1+\sqrt[3]{2}+\sqrt[3]{4}]^2, \text{ or } x=5,\ y=4,\ z=3,$$

which satisfy. Putting $n=3$, we find another set, and so on.

The method used in the solution of this problem is a particular case of a general principle applicable to the solution of many Diophantine questions.